

Welcome to the Putnam Seminar! Remember to join the HMC Google Group for the Putnam Seminar. This will keep you informed about the Seminar and the Putnam Exam on Saturday, December 4, 2021. To do this, send an email to: putnam-1+subscribe@g.hmc.edu

Putnam Practice Problems # 3

Tuesday, September 21, 2021

Sets and Counting

C1: How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40} , 20^{30} ? (Putnam, 1983)

C2: Arrange the integers $1, 2, \dots, n$ consecutively around a circle. Now, remove the number 2 and proceed by removing every other number until only one number remains. So, for example, if $n = 5$ the numbers removed are 2, 4, 1, 5 and 3 is the last number remaining.

(a) What is the last number remaining if $n = 2^p$?

(b) What is the last number remaining if $n = 2^p + 1$? (Adapted from Larson)

C3: Consider n distinct points in the plane, not all collinear. Prove that they determine at least n distinct lines. (Remember, a collection of points are collinear if they lie on the same line and two points determine the line that passes through them). (Newman)

C4: Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that $A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

(Putnam, 1985)

And for a little bit of variety...

C5: A deck of 52 playing cards is sitting on a table in a dark room. You are told that all the spades are face up and that all the remaining cards are face down. Your challenge is to arrange the cards into two piles each with the same number of cards face up without turning the lights on. How do you do it?

Hint: There are at least two solutions.

(A former Car Talk Puzzler)

Hints:

1. Draw a Venn diagram – what regions can the numbers occupy?
2. (a) Try a few examples and use induction. (b) Is this related to (a)?
3. What happens if every triple of points are collinear?
4. Place the numbers $\{1, 2, 3, \dots, 10\}$ into the appropriate Venn Diagram.