Welcome to the Putnam Seminar! Remember to join the HMC Google Group for the Putnam Seminar. This will keep you informed about the Seminar and the Putnam Exam on Saturday, December 4, 2021. To do this, send an email to: `putnam-l+subscribe@g.hmc.edu`

Putnam Practice Problems # 5

Tuesday, Oct 5, 2021

Find a Pattern

E1: Suppose $x + y = 1$ and $xy = -1$

(a) Find $x^2 + y^2$.
(b) Find $x^3 + y^3$.
(c) Find $x^{10} + y^{10}$.

E2: (a) Let S be the set of all numbers in the sequence $1, 2, 3, ...100$ that have no 0 digits. Let T be the set of all numbers in the sequence $1, 2, 3, ..., 1000$ that have exactly one digit that is 0. Show that the total number of digits in S is the same as the size of T.

(b) Show that for every positive integer n, the total number of digits in the sequence $1, 2, 3, ..., 10^n$ is equal to the total number of zero digits in the sequence $1, 2, 3, .., 10^{n+1}$. (Konhauser, Velleman, Wagon)

E3: Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$. For example, if $n = 4$, there are four ways: 4, 2+2, 1+1+2, 1+1+1+1. (Putnam, 2003)

E4: Greedy Pirates: You have 1000 pirates, who are all extremely greedy, heartless, and perfectly rational. They're also aware that all the other pirates share these characteristics. They're all ranked by the order in which they joined the group, from pirate one down to pirate one thousand.

They've stumbled across a huge horde of treasure, and they have to decide how to split it up. Every day they will vote to either kill the lowest ranking pirate, or split the treasure up evenly among the surviving pirates. If 50% or more of them vote to split it, the treasure gets split. Otherwise, they kill the lowest ranking pirate and repeat the process until half or more of the pirates decide to split the treasure.

The question, of course, is at what point will the treasure be split, and what will the precise vote be? (Newheiser & Wu)

And for a little bit of variety...

E5: Dissect a square into n isosceles right triangles of *different* sizes. How small can n be? (Hess)

Hints:

1. Can you use xy and $x + y$ to create $x^2 + y^2$? How about $x^3 + y^3$?
2. Find a one-to-one correspondence...
3. Find a pattern and use induction.
4. Again, work out a few of the small numbers and make a conjecture.