Math 93 Prof. Omar

Fall, 2021

Putnam TNG #1 – Probability

The Putnam TNG seminar assumes that you have tried to work the problems in advance of the seminar and have solved at least one problem that you are willing to present in class.

These problems will be discussed on 9/14/21.

A1: Initially Ashley has A dollars and Bob has B dollars. They play a fair game where after each round the loser gives the winner a dollar. They continue this until one of the players goes broke. What is the probability that Ashley wins all the money?

A2: Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside C? (Putnam, 1985)

A3: A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers. (Putnam, 1989)

A4: Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express your answer in the form $r + s\pi$, where r and s are rational numbers. (Putnam, 1993)

A5: (a) Three points are chosen at random on a circle. What is the probability that the center of the circle lies inside the triangle whose vertices are at the three points?

(b) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.) (Putnam, 1992)

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A6: Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S? (Putnam, 2006)

A7: Can you design a pair of cubical dice such that each face has a positive integer number of spots, and the probability that the sum of the spots on the dice when they are rolled is the same as for a normal pair of dice (where the number of spots on the faces are $1, 2, 3, 4, 5$ and 6), yet the dice are not normal? You should of course assume the dice are fair (meaning each face has a probability of $1/6$ of appearing).

A8: We throw N standard dice (with sides labeled $1, 2, 3, 4, 5, 6$ which appear with equal probability). What is the probability that the sum of the values appearing on the dice is divisible by 5? (IMC 1999)