Math 93 Prof. Omar

Tues Nov 2, 2021

Putnam TNG #7 – Geometry

G1: Find the positive value of m such that the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the x-axis, and the line $y = 2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the y-axis, and the line $y = mx$. (Putnam, 1994)

G2: Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position of s. (Putnam, 1998)

G3: A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC? (Putnam, 1997)

G4: Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle. (Putnam, 1996)

G5: Let A, B, C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC. (Putnam, 1998)

G6: Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists. (Putnam, 1998)

G7: Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY. (Putnam, 1996)

G8: Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals T, and O is the center of the circle. Determine the least upper bound of $(s_1 - s_2)/d$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained. (Putnam, 1989)